Generalised Asymmetric Quantum Cloning Machines

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We study machines that take N identical replicas of a pure qudit state as input and output a set of M_A clones of a given fidelity and another set of M_B clones of another fidelity. The trade-off between these two fidelities is investigated, and numerous examples of optimal $N \to M_A + M_B$ cloning machines are exhibited using a generic method. A generalisation to more than two sets of clones is also discussed. Finally, an optical implementation of some such machines is proposed. This paper is an extended version of [xxx.arxiv.org/abs/quant-ph/0411179].

I. INTRODUCTION

The no-cloning theorem [1] states that it is in general impossible to perfectly clone the state of a quantum system. This no-go result is a typical feature of quantum information, and is deeply related to quantum error correction [2], or the impossibility of super-luminal signalling [3]. But the no-cloning theorem can also be turned to a valuable resource as demonstrated by quantum cryptography [4].

Although perfect cloning is forbidden by quantum mechanics, it is of fundamental importance to analyse how well we can approximately copy a quantum state into several quantum systems in order to understand how quantum information distributes. The simplest instance of this problem is the duplication of the pure state of a qubit, and has been considered in [5] (a qubit is a two-level quantum system). Many generalisations and variants have followed. Among them are the issue of producing M clones of a qubit state from N input replicas [6] and its generalisation to qudits (d-level quantum systems) [7, 8], cloning from non identical input states [9], or non-universal cloning [10]. Cloning of continuous variable systems has also been considered [11].

This paper deals with universal $N \to M_A + M_B + \ldots$ asymmetric cloning of pure qudit states. That is, we are given N identical replicas of an unknown pure qudit state, and we want to produce M_A approximate clones all with a same fidelity, say F^A, M_B clones, all with a same fidelity F^B, ... We are interested in the trade-off between the qualities of the various sets of clones. Such machines have already been examined in [12], but only $1 \to 1 + 1$ cloning was considered, and optimality was only proven in the case of qubits. $N \to M_A + M_B$ cloning machines have at least two interesting applications. First, in the case where $M_A = N$ and $M_B \to \infty$, we get a machine that allows to study the trade-off between the acquisition of knowledge about the state of a quantum system and the disturbance undergone by this system. Second, some $N \to M_A + M_B$ cloning machines have been proven to be a useful tool when investigating the security of some quantum key distribution schemes [13, 14, 15].

Here is a summary of our results.

- A generic method to get optimal $N \to M_A + M_B$ cloning machines of pure qudit states is prescribed for any values of N, M_A, M_B, d, and a 'natural' conjecture of what cloning machines this method should always produce is proposed . This conjecture has been supported by all the cases we have examined. These results are essentially the content of Section II.

- With the aforementioned method, optimal asymmetric cloning of qubits has been investigated in the following cases: $1 \to 1 + n$ cloning of qubits (see section III C) and $n \to n + 1$ cloning of qubits (see Section III B).

- Optimal $1 \to 1 + 1 + 1$ cloning of qubits is analysed (Section III E).

- A probabilistic implementation of some such asymmetric machines is proposed. (Section IV).

II. OPTIMAL ASYMMETRIC CLONING MAPS

Before going into the details of cloning maps, let us sketch the idea behind the subsequent analysis. Let us consider the simplest case of $1 \to 2$ universal cloning of qubits. Any one-qubit state can be represented by a unit vector **n** on the surface of a sphere, the so-called Bloch vector on the Bloch sphere. The effect of an optimal symmetric cloning machine is to produce two output clones with reduced Bloch vectors η**n** ($\eta < 1$). That is, the cloning machine merely shrinks the input Bloch vector, but doesn't affect its orientation. This "isotropy" property results from the fact that no state is preferred by an optimal universal cloning machine. Therefore the quality of the cloning process is completely

characterised by the shrinking factor η. As demonstrated in [7], a similar argument applies for $N \to M$ cloning of qudits: the quality of the clones is fully characterised by some quantity ω, and the whole problem of optimal cloning amounts to extremise this quantity. As we will observe below, this argument still applies for *asymmetric* cloning. Consider $N \to M$ cloning and let $M_A + M_B = M$ denote a partition of M. If we require M_A clones to be of the same quality, and M_B clones to be of the same quality, the quality of the first (respectively second) set of clones is fully characterised by some quantity ω^A (respectively ω^B). Therefore, from an algebraic point of view, the main problem posed by optimal asymmetric cloning is to find some tight relation between ω^A and ω^B.

The object of this section is threefold: (i) to cast the cloning problem into purely algebraic terms, (ii) to give a quantitative relation between the qualities of the two sets of clones, (iii) to show how to get optimal cloning machines.

A. Cloning maps and figures of merit

An asymmetric cloning device would be an operation taking N identical replicas of an unknown state and outputting $M = M_A + M_B$ approximate clones. In Schrödinger picture, this operation is described by a trace-preserving completely positive map [16]

$$T' : \mathcal{B}(\mathcal{H}^{\otimes N}) \to \mathcal{B}(\mathcal{H}^{\otimes M}).$$

$\mathcal{H} = \mathbf{C}^d$ denotes the Hilbert space of a single qudit, $\mathcal{H}^{\otimes N}$ its N-fold tensor product, and $\mathcal{B}(\mathcal{H}^{\otimes N})$ denotes the space of bounded operators over $\mathcal{H}^{\otimes N}$. T' makes states evolve and leaves operators invariant . Equivalently, cloning can be described in Heisenberg picture by a unital ($T(\mathbf{1}) = \mathbf{1}$) completely positive map $T : \mathcal{B}(\mathcal{H}^{\otimes M}) \to \mathcal{B}(\mathcal{H}^{\otimes N})$. T makes operators evolve and states are now left invariant. The two pictures are equivalent and are related by the identity:

$$\mathrm{tr}(\rho T(\mathcal{O})) = \mathrm{tr}(\mathcal{O} T'(\rho)), \tag{1}$$

for all operator $\mathcal{O} \in \mathcal{B}(\mathcal{H}^{\otimes M})$ and for all density operator $\rho \in \mathcal{B}(\mathcal{H}^{\otimes N})$. Note that since all input systems are of the form $P_\psi^{\otimes N} \equiv |\psi\rangle\langle\psi|^{\otimes N}$, $|\psi\rangle \in \mathcal{H}$, we can without loss of generality consider the input Hilbert space to be only the symmetric subspace of $\mathcal{H}^{\otimes N}$: $\mathcal{H}_+^{\otimes N}$. So, cloning can be described by a unital cp-map

$$T : \mathcal{B}(\mathcal{H}^{\otimes M}) \to \mathcal{B}(\mathcal{H}_+^{\otimes N}). \tag{2}$$

There are essentially two options to quantify the quality of the clones. The first one consists in considering global figures of merit. That is if ψ denotes the input state to clone, one can consider the quantity

$$
\begin{aligned}
F_{\mathrm{all}}^A(T) &= \min_\psi \langle\psi^{\otimes M_A}| \, \mathrm{Tr}_{\mathcal{H}^{\otimes M_B}} T'(P_\psi^{\otimes N})|\psi^{\otimes M_A}\rangle \\
&= \min_\psi \langle\psi^{\otimes N}|(T(P_\psi^{\otimes M_A} \otimes \mathbf{1}^{\otimes M_B})|\psi^{\otimes N}\rangle,
\end{aligned} \tag{3}
$$

to quantify the first set of M_A clones, where $\mathrm{Tr}_{\mathcal{H}^{\otimes M_B}}$ denotes the partial trace over the second set of clones. A similar expression holds for the global fidelity of the second set of clones, $F_{\mathrm{all}}^B(T)$.

Alternatively, one can consider single-copy fidelities:

$$
\begin{aligned}
F^A(T) &= \min_{\psi \in \mathcal{H}} \min_{1 \le k \le M_A} \langle\psi| \, \mathrm{Tr}_k T'(|\psi^{\otimes N}\rangle\langle\psi^{\otimes N}|)|\psi\rangle \\
&= \min_{\psi \in \mathcal{H}, k} \langle\psi^{\otimes N}|(T(\mathbf{1}^{\otimes k-1} \otimes |\psi\rangle\langle\psi| \otimes \mathbf{1}^{\otimes M_A + M_B - k})|\psi^{\otimes N}\rangle,
\end{aligned} \tag{4}
$$

for the first set of M_A clones. In this expression, Tr_k denotes the partial trace over all clones but the k-th in the first set. A similar expression holds for the single-clone fidelity of the second set of clones, $F^B(T)$.

Even in the symmetric case, it is highly non-trivial to prove that a quantum cloning machine which is optimal for one figure of merit is optimal for the other [7]. In this work we choose to consider single-clone fidelities because it is more relevant than the global fidelities when one considers connections of cloning with other tasks, such as state estimation (see Section III C), or quantum cryptography [13].

Considering the first set of clones, it is obvious that $F^A(T)$ lies between the fidelity obtained when preparing the clones in a random state, and the fidelity of the optimal $N \to M_A$ symmetric cloning machine $F_{\mathrm{sym}}^A(N, M_A)$. The problem of finding optimal $N \to M_A + M_B$ asymmetric cloning can now be clearly formulated: To maximise $F^B(T)$ for a given value of $F^A(T)$ between $1/d$ (the fidelity of a machine producing outputs in a random state) and $F_{\mathrm{sym}}^A(N, M_A)$ (the fidelity of an optimal $N \to M_A$ symmetric machine).

Before proceeding any further, let us fix some notations. These notations are similar to those in the paper of Keyl and Werner [7], so as to make the connection between their work and ours as transparent as possible. U(d) will denote the (compact Lie) group of unitary $d \times d$ matrices, and SU(d) will denote the subgroup of U(d) of elements whose

determinant equals 1. The representations of U(d) will be denoted π_α. Of particular importance are: (i) the natural representation, i.e. the representation given by the elements of the group themselves, which we will simply denote as π, (ii) the N-fold tensor product of π, $\pi^{\otimes N}$ and (iii) the irreducible representation given by the restriction of $\pi^{\otimes N}$ to the symmetric subspace $\mathcal{H}_+^{\otimes N}$ of $\mathcal{H}^{\otimes N}$: π_N^+. Thus, if $\psi_0 \in \mathbf{C}^d$ denotes some fiducial state, the set of input states to clone reads

$$\{\pi_N^+(u)|\psi_0\rangle^{\otimes N} : u \in \mathrm{U}(d)\}.$$

In the special case $d = 2$, as is well known, the irreducible representations are labelled by half-integer positive numbers. Accordingly, these irreducible representations will be denoted $\pi_0, \pi_{1/2}$, etc. u(d) will denote the Lie algebra associated to U(d), i.e. u(d) is the Lie algebra of antihermitian matrices, and su(d), the algebra consisting of traceless elements of u(d). The irreducible representations of u(d) will be denoted $\partial\pi_\alpha$. Let X denote an element of u(d) and let $\{e^{tX} \in \mathrm{U}(d) : t \in \mathbf{R}\}$ denote the associated one-parameter subgroup of U(d). We have

$$\pi_\alpha(e^{tX}) = e^{t\partial\pi_\alpha(X)}.$$

Also note that the representation of $\partial\pi^{\otimes M}$ can be expressed very simply from the natural representation $\partial\pi$:

$$\partial\pi^{\otimes M}(X) = \sum_{i=1}^M \mathbf{1}^{\otimes i-1} \otimes \partial\pi(X) \otimes \mathbf{1}^{\otimes M-i}.$$

B. A natural conjecture

The optimal symmetric cloning map reads (in Schrödinger picture) [8]:

$$T'_{\mathrm{sym}} : \mathcal{B}(\mathcal{H}_+^{\otimes N}) \to \mathcal{B}(\mathcal{H}^{\otimes M}) : \rho^{\otimes N} \to \frac{d[N]}{d[M]} S_M(\rho^{\otimes N} \otimes \mathbf{1}^{\otimes M-N}) S_M, \tag{5}$$

where $d[N] = \dim\mathcal{H}_+^{\otimes N}$ (the constant $d[N]/d[M]$ ensures that the map is trace-preserving). S_M is the projector onto the symmetric subspace $\mathcal{H}_+^{\otimes M}$. The interpretation of this map is quite intuitive: $M - N$ states containing no (quantum) information are appended to the input, and the resulting state is symmetrised.

Note that the representation $\pi^{\otimes M}$ decomposes as $\pi^{\otimes M} = \pi_M^+ \oplus \pi_{\mathrm{rest}}$, where π_{rest} is some representation containing no representation equivalent to π_M^+ and acting on some space $\mathcal{H}_{\mathrm{rest}}$ [17]. Accordingly, we have $\mathbf{1}^{\otimes M} = S_M + \mathbf{1}_{\mathrm{rest}}$, where $\mathbf{1}_{\mathrm{rest}}$ denotes the identity over $\mathcal{H}_{\mathrm{rest}}$. Thus, what the cp-map (5) suggests is that optimality is achieved by keeping only the component of the decomposition corresponding to the symmetric subspace.

In the case of asymmetric cloning, we have to consider the decomposition $\pi^{\otimes M_A} \otimes \pi^{\otimes M_B} \approx (\pi_{M_A}^+ \otimes \pi_{M_B}^+) \oplus (\pi_{M_A}^+ \otimes \pi_{\mathrm{rest'}}) \oplus (\pi_{\mathrm{rest}} \otimes \pi_{M_B}^+) \oplus (\pi_{\mathrm{rest}} \otimes \pi_{\mathrm{rest'}})$. According to this decomposition, we have $\mathbf{1}^{\otimes M} = \mathbf{1}^{\otimes M_A} \otimes \mathbf{1}^{\otimes M_B} = S_{M_A} \otimes S_{M_B} + S_{M_A} \otimes \mathbf{1}_{\mathrm{rest'}} + \mathbf{1}_{\mathrm{rest}} \otimes S_{M_B} + \mathbf{1}_{\mathrm{rest}} \otimes \mathbf{1}_{\mathrm{rest'}}$.

Our conjecture is that only the piece $\pi_{M_A}^+ \otimes \pi_{M_B}^+$ should be considered in this decomposition. More precisely, let $\pi_{M_A}^+ \otimes \pi_{M_B}^+ \approx \oplus_\gamma \pi_\gamma$ denote the decomposition of $\pi_{M_A}^+ \otimes \pi_{M_B}^+$ into irreducible components, and let E_γ denote the projector associated to the irreducible component π_γ. We conjecture that optimal asymmetric cloning machines should be of the form:

$$T'_{\mathrm{asym}} : \mathcal{B}(\mathcal{H}_+^{\otimes N}) \to \mathcal{B}(\mathcal{H}^{\otimes M}) : \rho^{\otimes N} \to V^*(\rho^{\otimes N} \otimes \mathbf{1}^{\otimes M-N}) V, \tag{6}$$

where V is a linear combination of projectors E_γ. This conjecture is supported by *all* asymmetric machines we have considered.

C. How to get optimal asymmetric cloning maps

We now describe a general recipe to get optimal asymmetric cloning machines. This section summarises the results obtained in Appendix A. $N \to M_A + M_B$ cloning of qudits is achieved by a cp-map $T : \mathcal{B}(\mathcal{H}^{\otimes M}) \to \mathcal{B}(\mathcal{H}_N^+)$, which decomposes as

$$T = \sum_{\alpha_1 \in D(\pi^{\otimes M_A})} \sum_{\alpha_2 \in D(\pi^{\otimes M_B})} \sum_{\beta \in D_N(\pi_{\alpha_1} \otimes \pi_{\alpha_2})} r(\alpha_1, \alpha_2, \beta) \, T(\alpha_1, \alpha_2, \beta), \tag{7}$$

where we define $D(\pi_Z) = \{a : \pi_a \subset \pi_Z\}$ and $D_N(\pi_Z) = \{a : \pi_N^+ \subset \pi_Z \otimes \pi_a\}$. The quantities $r(\alpha_1, \alpha_2, \beta)$ satisfy

$$r(\alpha_1, \alpha_2, \beta) \geq 0, \qquad \sum_{\alpha_1, \alpha_2, \beta} r(\alpha_1, \alpha_2, \beta) = 1. \tag{8}$$

The (single-clone) fidelities are essentially fixed by two quantities, $\omega^A(T)$ and $\omega^B(T)$, analogous to the shinrking factor η discussed at the beginning of Sect.II. We have

$$F^A(T) = \frac{1}{d}(1 + \frac{N}{M_A}\omega^A(T)(d-1)), \tag{9}$$

$$F^B(T) = \frac{1}{d}(1 + \frac{N}{M_B}\omega^B(T)(d-1)). \tag{10}$$

The quantities $\omega^A(T)$ and $\omega^B(T)$ are decomposed according to the convex decomposition of T into irreducible summands:

$$\omega^A(T) = \sum_{\alpha_1 \in D(\pi^{\otimes M_A})} \sum_{\alpha_2 \in D(\pi^{\otimes M_B})} \sum_{\beta \in D_N(\pi_{\alpha_1} \otimes \pi_{\alpha_2})} r(\alpha_1, \alpha_2, \beta)\, \omega^A(T(\alpha_1, \alpha_2, \beta)), \tag{11}$$

$$\omega^B(T) = \sum_{\alpha_1 \in D(\pi^{\otimes M_A})} \sum_{\alpha_2 \in D(\pi^{\otimes M_B})} \sum_{\beta \in D_N(\pi_{\alpha_1} \otimes \pi_{\alpha_2})} r(\alpha_1, \alpha_2, \beta)\, \omega^B(T(\alpha_1, \alpha_2, \beta)). \tag{12}$$

The quantities $\omega^A(T(\alpha_1, \alpha_2, \beta)$ are given by

$$\omega_{\alpha_1, \alpha_2, \beta}^A = \frac{1}{2} \sum_{a \in D(\pi_{\alpha_2} \otimes \pi_\beta)} |\lambda_a|^2 \{1 + \frac{C_2(\pi_{\alpha_1}) - C_2(\pi_a)}{C_2(\pi_N^+)}\}, \tag{13}$$

where $C_2(\pi_a)$ denotes the Casimir number associated to the irreducible representation π_a. The quantities λ_a satisfy

$$\sum_a |\lambda_a|^2 = 1. \tag{14}$$

Similarly,

$$\omega_{\alpha_1, \alpha_2, \beta}^B = \frac{1}{2} \sum_{b \in D(\pi_{\alpha_1} \otimes \pi_\beta)} |\mu_b|^2 \{1 + \frac{C_2(\pi_{\alpha_2}) - C_2(\pi_b)}{C_2(\pi_N^+)}\}, \tag{15}$$

with $\sum_b |\mu_b|^2 = 1$.

The quantities λ_a and μ_b are related by

$$\mu_b = \frac{1}{\dim\mathcal{H}_+^{\otimes N}} \sum_a \lambda_a \operatorname{tr} W_b^* V_a. \tag{16}$$

V_a and W_b are intertwining operators defined as follows. Let X denote an auxiliary space supporting the representation β. Consider the decomposition theory of $\pi_{\alpha_1}^A \otimes \pi_{\alpha_2}^B \otimes \pi_\beta^X \approx \pi_N^+ \oplus \pi_{\text{rest}}$, where π_{rest} contains no copy of π_N^+. V_a and W_b are defined as the unique isometries such that

$$(\pi_{\alpha_1}^A \otimes \pi_a^{BX})V_a = V_a \pi_N^+, \tag{17}$$

$$(\pi_{\alpha_2}^B \otimes \pi_b^{AX})W_b = W_b \pi_N^+. \tag{18}$$

It is now clear that the problem of finding optimal cloning machines is a constrained optimisation problem: We have to maximize $\omega^B(T)$ for a fixed value of $F^A(T)$ taking into account the constraint (8)-(14)-(16).

In the next section, we apply the recipe just described to treat some concrete examples.

III. SOME ASYMMETRIC CLONING MACHINES

A. The simplest example: $1 \to 1 + 1$ cloning of qubits

Although $1 \to 1 + 1$ cloning machines of qubits have been extensively studied [12], it is instructive to revisit them in order to illustrate as simply as possible the foregoing analysis.

Adopting the standard convention of denoting irreducible representations of SU(2) by half-integer numbers, we have here $D(\pi^{\otimes M_A}) = D(\pi^{\otimes M_B}) = \{1/2\}$, and $D_1(1/2 \otimes 1/2) = \{\beta : 1/2 \subset 1/2 \otimes 1/2 \otimes \beta\} = \{3/2, 1/2\}$, where use was made of the Clebsch-Gordan series

$$j_1 \otimes j_2 \approx |j_1 - j_2| \oplus \ldots \oplus (j_1 + j_2),$$

for the decomposition of the tensor product of two irreducible representations. Accordingly,

$$\omega^A(T) = r(1/2, 1/2, 3/2)\omega^A(T(1/2, 1/2, 3/2)) + r(1/2, 1/2, 1/2)\omega^A(T(1/2, 1/2, 1/2)), \tag{19}$$
$$\omega^B(T) = r(1/2, 1/2, 3/2)\omega^B(T(1/2, 1/2, 3/2)) + r(1/2, 1/2, 1/2)\omega^B(T(1/2, 1/2, 1/2)). \tag{20}$$

With $C_2(j) = j(j+1)$, we have:

$$\omega^A(T(1/2, 1/2, 3/2)) = \omega^B(T(1/2, 1/2, 3/2)) = \frac{1}{2}(1 + \frac{C_2(1/2) - C_2(1)}{C_2(1/2)}) = -1/3.$$

So the map $T(1/2, 1/2, 3/2)$ is useless for cloning, since the quantities ω^A and ω^B it yields are worse than for a map which would consit of preparing the clones in a random state ($\omega^A_{\text{random}} = \omega^B_{\text{random}} = 0$). Let us consider the other map, $T(1/2, 1/2, 1/2)$. We have

$$\omega^A(T(1/2, 1/2, 1/2)) = \frac{1}{2}(|\lambda_0|^2(1 + \frac{C_2(1/2) - C_2(1/2)}{C_2(0)}) + |\lambda_1|^2(1 + \frac{C_2(1/2) - C_2(1)}{C_2(1/2)})$$
$$= \frac{1}{2}(2|\lambda_0|^2 - \frac{2}{3}|\lambda_1|^2). \tag{21}$$

Similarly,

$$\omega^B(T(1/2, 1/2, 1/2)) = \frac{1}{2}(2|\mu_0|^2 - \frac{2}{3}|\mu_1|^2). \tag{22}$$

Let us now work out the relation between the coefficients λ_0, λ_1 and μ_0, μ_1. In turn, this relation will give us the trade-off between the fidelities of the clone A and the clone B. This relation involves four isometric interwiners: V_0, V_1, W_0 and W_1. Explicitly, we have

$$V_0 = |(1/2^A 0^{BX})1/2, m\rangle\langle 1/2, m| = C^{(1/2s)(0t)}_{(1/2m)} C^{(1/2u)(1/2v)}_{(0t)} |1/2, s\rangle_A |1/2, u\rangle_B |1/2, v\rangle_X \langle 1/2, m|, \tag{23}$$

$$V_1 = |(1/2^A 1^{BX})1/2, m\rangle\langle 1/2, m| = C^{(1/2s)(1t)}_{(1/2m)} C^{(1/2u)(1/2v)}_{(1t)} |1/2, s\rangle_A |1/2, u\rangle_B |1/2, v\rangle_X \langle 1/2, m|, \tag{24}$$

$$W_0 = |(1/2^B 0^{AX})1/2, m\rangle\langle 1/2, m| = C^{(1/2s)(0t)}_{(1/2m)} C^{(1/2u)(1/2v)}_{(0t)} |1/2, u\rangle_A |1/2, s\rangle_B |1/2, v\rangle_X \langle 1/2, m|, \tag{25}$$

$$W_1 = |(1/2^B 1^{AX})1/2, m\rangle\langle 1/2, m| = C^{(1/2s)(1t)}_{(1/2m)} C^{(1/2u)(1/2v)}_{(1t)} |1/2, u\rangle_A |1/2, s\rangle_B |1/2, v\rangle_X \langle 1/2, m|, \tag{26}$$

where $C^{(**)(**)}_{(**)}$ denote Clebsch-Gordan coupling coefficients and sum over repeated indices is understood. In this expression, $|j, m\rangle$ denote elements of an orthonormal basis for a spin-j representation of SU(2), and $|(j_1j_2)j, m\rangle$ denote elements of a basis for the irreducible spin-j representation contained in $j_1 \otimes j_2$. From Eqs(23)-(26), one can verify that $\operatorname{tr} W_1^* V_0 = \sqrt{3}$ and $\operatorname{tr} W_1^* V_1 = -1$. We can, without loss of optimality, assume that $\lambda_0, \lambda_1, \mu_0$ and μ_1 are real. Using $|\lambda_0|^2 + |\lambda_1|^2 = 1$ and $|\mu_0|^2 + |\mu_1|^2 = 1$. We get

$$\omega^A(1/2, 1/2, 1/2) = 1 - \frac{4}{3}\lambda_1^2, \tag{27}$$
$$\omega^B(1/2, 1/2, 1/2) = 1 - \frac{4}{3}\mu_1^2 = 1 - \frac{4}{3}(\frac{\sqrt{3}}{2}\sqrt{1 - \lambda_1^2} - \frac{1}{2}\lambda_1)^2. \tag{28}$$

The corresponding fidelities are

$$F^A = (1 + \omega^A(1/2, 1/2, 1/2))/2, \tag{29}$$
$$F^B = (1 + \omega^B(1/2, 1/2, 1/2))/2. \tag{30}$$

On Fig.1, we have plotted the locus $\{(F^A(\lambda_1), F^B(\lambda_1))\}$. One readily checks that this locus of couples of fidelities correspond with the results in [12].

B. $n \to n+1$ **cloning of qubits**

We now solve the $2 \to 2+1$ case and give the $n \to n+1$ conjectured optimal fidelities. The obtained figures of merit agree with the expected values at the limiting points.

Let us start with the $2 \to 2+1$ derivation. First, let us observe that $D(\pi_{1/2}^{\otimes 2}) = \{\pi_0, \pi_1\}$, $D(\pi_{1/2}) = \{\pi_{1/2}\}$, $D_2(\pi_0 \otimes \pi_{1/2}) = \{\pi_{1/2}, \pi_{3/2}\}$, $D_2(\pi_1 \otimes \pi_{1/2}) = \{\pi_{1/2}, \pi_{3/2}, \pi_{5/2}\}$. Therefore, according to Sect.II C, the optimal map we are looking for can be decomposed as a convex sum of five maps as:

$$\begin{align}
T = \quad & r(0,1/2,1/2)T(0,1/2,1/2) + r(0,1/2,3/2)T(0,1/2,3/2) + r(1,1/2,1/2)T(1,1/2,1/2) + \tag{31}\\
& r(1,1/2,3/2)T(1,1/2,3/2) + r(1,1/2,5/2)T(1,1/2,5/2). \tag{32}
\end{align}$$

Consider the map $T(0,1/2,1/2)$. This map is characterised by an intertwining operator V satisfying

$$V\pi_1 = (\pi_0^A \otimes \pi_{1/2}^B \otimes \pi_{1/2}^X)V. \tag{33}$$

There exists a (Clebsch-Gordan) matrix C such that $\pi_{1/2}^B \otimes \pi_{1/2}^X = C^*(\pi_0^{BX} \oplus \pi_1^{BX})C$. With $V' = CV$, we thus have

$$V\pi_1 = (\pi_0^A \otimes (\pi_0^{BX} \oplus \pi_1^{BX}))V. \tag{34}$$

From Eq.(13), one then sees that

$$\omega^A(0,1/2,1/2) = \frac{1}{2}(1 + \frac{C_2(\pi_0) - C_2(\pi_1)}{C_2(\pi_1)}) = 0. \tag{35}$$

One can also compute that

$$\omega^B(0,1/2,1/2) = 1/2. \tag{36}$$

Performing the same analysis for the map $T(0,1/2,3/2)$, one finds that $\omega^A(0,1/2,3/2) = 0$ and $\omega^B(0,1/2,3/2) = -1/4$. Since a trivial cp-map that merely prepares clones in a random state achieves $\omega^A = \omega^B = 0$, we see that this latter map is useless for cloning. The map $T(1,1/2,5/2)$ also turns out to be useless because $\omega^A(1,1/2,5/2) = -1/2$ and $\omega^B(1,1/2,5/2) = -1/4$. Consider now the map $T(1,1/2,1/2)$. The intertwiner that characterises this map (and that we will again denote V) satisfies

$$V\pi_1 = (\pi_1^A \otimes \pi_{1/2}^B \otimes \pi_{1/2}^X)V. \tag{37}$$

Again there exists a (Clebsch-Gordan) matrix C such that $V' = CV$ satisifies

$$V'\pi_1 = (\pi_1^A \otimes (\pi_0^{BX} \oplus \pi_1^{BX}))V'. \tag{38}$$

The space of solutions for Eq.(38) is 2-dimensional: V' is a linear combination of an intertwiner V_0' between π_1 and $\pi_1^A \otimes \pi_0^{BX}$, and an intertwiner V_1' between π_1 and $\pi_1^A \otimes \pi_1^{BX}$. Accordingly, one finds that

$$\omega^A(1,1/2,1/2) = |\lambda(1,1/2,1/2,0)|^2\frac{1}{2}(1 + \frac{C_2(1) - C_2(0)}{C_2(1)}) + |\lambda(1,1/2,1/2,1)|^2\frac{1}{2}(1 + \frac{C_2(1) - C_2(1)}{C_2(1)}) \tag{39}$$

$$= |\lambda(1,1/2,1/2,0)|^2 + \frac{1}{2}|\lambda(1,1/2,1/2,1)|^2, \tag{40}$$

where $|\lambda(1,1/2,1/2,0)|^2 + |\lambda(1,1/2,1/2,1)|^2 = 1$ (because V is an isometry). Considering the second set of clones, on finds that

$$\omega^B(1,1/2,1/2) = \frac{1}{2}|\mu(1,1/2,1/2,1/2)|^2 - \frac{1}{4}|\mu(1,1/2,1/2,3/2)|^2, \tag{41}$$

where $|\mu(1,1/2,1/2,1/2)|^2 + |\mu(1,1/2,1/2,3/2)|^2 = 1$. A similar analysis of the map $T(1,1/2,3/2)$ shows that

$$\omega^A(1,1/2,3/2) = \frac{1}{2}(|\lambda(1,1/2,3/2,1)|^2 - |\lambda(1,1/2,3/2,2)|^2), \tag{42}$$

and where $|\lambda(1,1/2,3/2,1)|^2 + |\lambda(1,1/2,3/2,2)|^2 = 1$, and

$$\omega^B(1,1/2,3/2) = \frac{1}{2}|\mu(1,1/2,3/2,1/2)|^2 - \frac{1}{4}|\mu(1,1/2,3/2,3/2)|^2, \tag{43}$$

where $|\mu(1, 1/2, 3/2, 1/2)|^2 + |\mu(1, 1/2, 3/2, 3/2)|^2 = 1$. Extremising numerically, we have found that optimal asymmetric cloning machines are such that $r(1, 1/2, 1/2) = 1$. Adopting lighter notations, the fidelities can be written as

$$F^A = 1 - \frac{x^2}{4}, \quad F^B = 1 - \frac{(x + \sqrt{2}y)^2}{4}, \tag{44}$$

where $x^2 + y^2 = 1$. Accoding to Sect.IIC, optimal machines should be of the form (Schrdinger picture)

$$T_{\text{opt}} : \mathcal{B}(\mathcal{H}_+^{\otimes 2}) \rightarrow \mathcal{B}(\mathcal{H}^{\otimes 3}) : \rho^{\otimes n} \rightarrow (a^* E_{3/2} + b^* E_{1/2})(\rho^{\otimes 2} \otimes \mathbf{1})(a E_{3/2} + b E_{1/2}), \tag{45}$$

where $E_{3/2}$ and $E_{1/2}$ are the projectors onto the irreducible subspaces obtained from the decomposition $\pi_1 \otimes \pi_1 \approx \pi_{3/2} \oplus \pi_{1/2}$. Also, $2(2a^2 + b^2)/3 = 1$ because T_{opt} has to be trace-preserving. Note that the fidelities for the limiting cases are recovered, that is

$$F_1 = F_2 = 1 \iff F_3 = \frac{1}{2}$$

$$F_1 = F_2 = F_3 = F_{\text{sym}}(2 \rightarrow 3) = \frac{11}{12}$$

$$F_1 = F_2 = F_{\text{sym}}(1 \rightarrow 2) = \frac{5}{6} \iff F_3 = 1. \tag{46}$$

The previous computation strongly supports that the conjectured machines found by combining linearly the projectors $E_{(n-1)/2}$ and $E_{(n-1)/2}$ coming from the decomposition $\pi_{n/2} \otimes \pi_{1/2} \approx \pi_{(n-1)/2} \oplus \pi_{(n+1)/2}$. The corresponding fidelities are given by

$$F_1 = F_2 = \ldots = F_n = 1 - \frac{2}{n(n+2)} x^2 \qquad F_{n+1} = 1 - \frac{1}{2}\left(\sqrt{\frac{n}{n+2}} x - y\right)^2, \tag{47}$$

with $x^2 + y^2 = 1$.

These fidelities are depicted in figure 1 for $n = 1, \ldots, 3$, where only the relevant part of the curve is shown. The extreme cases are now

$$F_1 = F_2 = \ldots = F_n = 1 \iff F_{n+1} = \frac{1}{2} \iff y = 0$$

$$F_1 = F_2 = \ldots = F_{n+1} = F_S(n \rightarrow n+1) = \frac{n^2 + 3n + 1}{n^2 + 3n + 2} \iff y = \left(\sqrt{\frac{n}{2(n+1)}}\right)$$

$$F_1 = F_2 = \ldots = F_n = F_S(n-1 \rightarrow n) = \frac{n^2 + n - 1}{n(n+1)} \iff F_{n+1} = 1 \iff y = \left(\sqrt{\frac{n+2}{2(n+1)}}\right). \tag{48}$$

When $n \rightarrow \infty$, an unlimited number of copies of the initial state are available. Then, it is possible to completely determine it and prepare a new identical copy. Then $F_{n+1} \rightarrow 1$ as expected.
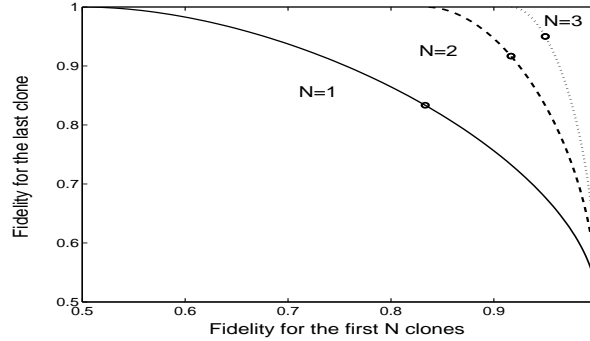

FIG. 1: Fidelities for the $n \rightarrow n + 1$ cloning machines. The curves for $n = 1, 2$ are known to be optimal. For $n > 2$ the curve is conjectured to be optimal. Dots on the curves correspond to symmetric machines.

C. $1 \to 1 + n$ cloning of qubits

The method described in Section II C has been applied to the case of qubits where one set, A, consists of one clone, and the other set, B, consists of n clones ($n \geq 2$).

The optimal fidelities are derived in Appendix B and are given by

$$F^A = 1 - \frac{2}{3}y^2, \qquad F^B = \frac{1}{2} + \frac{1}{3n}(y^2 + \sqrt{n(n+2)}xy), \tag{49}$$

where $x^2 + y^2 = 1$.

One can check that these fidelities correspond to those obtained using the conjectured form of the optimal cloning map.

Let us discuss these expressions. First of all, imposing $F^A = 1$ implies $F^B = 1/2$. This fact is consistent with the idea that in order to prepare a perfect clone, one has to take it from the input and not let it interact with any system [1]. Then, no quantum information is available to prepare the N supplementary clones and the best one can do is to prepare the N qubits in a completely random state, thus achieving a fidelity $1/2$.

Second, if one requires the clones B to have the fidelity of an optimal symmetric $1 \to n$ cloning machine, namely $F^B = \frac{2n+1}{3n}$, one finds that $y = \sqrt{\frac{n+2}{2n+2}}$, which gives $F^A = \frac{2n+1}{3(n+1)}$. Interestingly this fidelity is larger than $1/2$: in order to produce n optimal clones from one input, not all quantum information need be used and some quantum information remains to prepare a "good" $(n+1)$-th clone.

Let us now turn to the case of large n. It is well known that there are deep connections between cloning and state estimation [6, 16, 18]. In particular, for universal symmetric cloning, it appears that there is a correspondence between $n \to \infty$ cloning machines and state estimation devices [19]. Such a relation still holds in the asymmetric case. Following the lines of [19], one finds that, in the limit $n \to \infty$, asymmetric $1 \to 1 + n$ cloning machines interpolate between (trivial) machines leaving the quantum system unchanged, and a measuring device estimating destructively the input state. In the limit $n \to \infty$, Eqs. (49) become:

$$F^A = 1 - \frac{2}{3}y^2, \qquad F^{\text{meas}} = \frac{1}{2} + \frac{1}{3}y\sqrt{1 - y^2}, \tag{50}$$

where only the case $0 \leq y \leq 1/\sqrt{2}$ should be considered. One readily checks that the two extreme cases are found:(i) When $F^A = 1$, one finds $F^{\text{meas}} = 1/2$, which translates the fact that no information can be gained if the input state is unperturbed. (ii) The maximum value of F^{meas} is $2/3$, which is consistent with [20]. In that case, of course, $F^A = 2/3$ too. Between these two cases, the relations (50) express the trade-off between the acquisition of knowledge about the state of a quantum system and the disturbance undergone by this system. Actually, such a trade-off had been previously studied in [21], in the form of an inequality. So, our machine provides a concrete means to achieve measurements saturating this inequality.

D. $1 \to 1 + 1$ cloning of qudits

$1 \to 1 + 1$ asymmetric machines were first introduced in [12]. The main interest of such machines is that they are useful in assessing the security of quantum cryptographic protocols [22]. However, such machines were known to be optimal only in the case of qubits. In principle, we could apply the method presented in Section II C to prove the optimality of these. We did not perform such a calculation. Alternatively, one could prove the optimality of these asymmetric machines using the isomorphism between CP maps and positive semidefinite operators [23, 24]. Perhaps not surprisingly, one finds that optimal cloning machines *are* of the form (6). Under $\pi^{\otimes 2}$, $\mathcal{H}^{\otimes 2}$ decomposes as a $\frac{d(d+1)}{2}$-dimensional symmetric subspace $\mathcal{H}_2^+$ and a $\frac{d(d-1)}{2}$-dimensional anti-symmetric subspace $\mathcal{H}_2^-$:

$$\pi^{\otimes 2} \approx \pi_2^+ \oplus \pi_2^-, \qquad \mathcal{H}^{\otimes 2} \approx \mathcal{H}_2^+ \oplus \mathcal{H}_2^-, \qquad \mathbf{1}^{\otimes 2} = S_2 + A_2. \tag{51}$$

Let $\{|i\rangle, i = 0 \ldots d-1\}$ denote an orthonormal basis of $\mathcal{H} = \mathbf{C}^d$. Clearly,

$$S_2 = \frac{1}{2} \sum_{i,j=0}^{d-1} (|i\rangle\langle i| \otimes |j\rangle\langle j| + |i\rangle\langle j| \otimes |j\rangle\langle i|), \qquad A_2 = \frac{1}{2} \sum_{i,j=0}^{d-1} (|i\rangle\langle i| \otimes |j\rangle\langle j| - |i\rangle\langle j| \otimes |j\rangle\langle i|). \tag{52}$$

According to Eq.(6), in Schrödinger picture, the optimal cloning map is of the form

$$T'_{\text{opt}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^{\otimes 2}) : \rho \to (\alpha^* S_2 + \beta^* A_2)(\rho \otimes \mathbf{1})(\alpha S_2 + \beta A_2). \tag{53}$$

Since $T'_{\rm opt}$ should be trace-preserving, we have

$$\frac{d+1}{2}|\alpha|^2 + \frac{d-1}{2}|\beta|^2 = 1. \tag{54}$$

Since the map $T'_{\rm opt}$ is covariant, the fidelity is the same for all input state, and can be calculated with a particular state, say $|0\rangle\langle 0|$. Straightforwardly, we get:

$$F^A = {\rm tr}(|0\rangle\langle 0| \otimes \mathbf{1})T'_{\rm opt}(|0\rangle\langle 0|) = \frac{d+3}{4}|\alpha|^2 + \frac{d-1}{4}|\beta|^2 + \frac{d-1}{4}(\alpha^*\beta + \alpha\beta^*). \tag{55}$$

$$F^B = {\rm tr}(\mathbf{1} \otimes |0\rangle\langle 0|)T'_{\rm opt}(|0\rangle\langle 0|) = \frac{d+3}{4}|\alpha|^2 + \frac{d-1}{4}|\beta|^2 - \frac{d-1}{4}(\alpha^*\beta + \alpha\beta^*). \tag{56}$$

$$\tag{57}$$

Direct calculations show that these machines correspond to the universal asymmetric cloning machines of qudits introduced in [12].

E. $1 \to 1+1+1$ cloning of qubits

We now turn to asymmetric cloning machines with more than two sets of clones. The simplest example of such a cloning machine is a $1 \to 1+1+1$ cloning machine of qubits which we shall exhibit now. Following Sect.II C, our first task is to determine the representations π_β satisfying $\pi_{1/2}^{\otimes 3} \otimes \pi_\beta \supset \pi_{1/2}$. One directly finds $\pi_\beta = \pi_0, \pi_1, \pi_2$. Accordingly, optimal $1 \to 1+1+1$ cloning maps are of the form

$$T = r_0 T_0 + r_1 T_1 + r_2 T_2,$$

where $T_\beta : \mathcal{B}(\mathcal{H}_{1/2}^{\otimes 3}) \to \mathcal{B}(\mathcal{H}_{1/2}) : A \to V_\beta^*(A \otimes \mathbf{1}_\beta)V_\beta$, with $(\pi_{1/2}^{\otimes 3} \otimes \pi_\beta)V_\beta = V_\beta \pi_{1/2}$, and where $r_0, r_1, r_2 \geq 0, r_0 + r_1 + r_2 = 1$. The fidelities for the three clones are given by

$$F^A = \frac{1}{2}(1 + \omega^A) \qquad\qquad F^B = \frac{1}{2}(1 + \omega^B) \qquad\qquad F^C = \frac{1}{2}(1 + \omega^C)$$

where

$$\omega^A = r_0\omega_0^A + r_1\omega_1^A + r_2\omega_2^A, \qquad \omega^B = r_0\omega_0^B + r_1\omega_1^B + r_2\omega_2^B, \qquad \omega^C = r_0\omega_0^C + r_1\omega_1^C + r_2\omega_2^C.$$

Let us work out expressions for $\omega_0^A, \omega_0^B, \omega_0^C$. Let us consider the operator V_0. It satisfies

$$(\pi_{1/2}^A \otimes \pi_{1/2}^B \otimes \pi_{1/2}^C \otimes \pi_0^X)V_0 = V_0 \pi_{1/2}.$$

Considering the reduction order $A \otimes B \otimes C \otimes X \to A \otimes BC \otimes X \to A \otimes BCX$, one finds that V_0 is of the form $V_0 = \lambda_0 E_{00} + \lambda_1 E_{01}$, where $|\lambda_0|^2 + |\lambda_1|^2 = 1$, and where

$$E_{00} = C_{(1/2m)}^{(1/2u)(00)}C_{(00)}^{(1/2v)(1/2w)}|1/2u\rangle_A|1/2v\rangle_B|1/2w\rangle_C|00\rangle_X\langle 1/2m|$$

$$E_{01} = C_{(1/2m)}^{(1/2u)(1k)}C_{(1k)}^{(1l)(00)}C_{(1l)}^{(1/2v)(1/2w)}|1/2u\rangle_A|1/2v\rangle_B|1/2w\rangle_C|1l\rangle_X\langle 1/2m|.$$

Using Eq.(13), one finds that

$$\omega_0^A = 1 - \frac{4}{3}\lambda_1^2$$

Similarly, considering the reduction order $A \otimes B \otimes C \otimes X \to B \otimes AC \otimes X \to B \otimes ACX$ (resp.$A \otimes B \otimes C \otimes X \to C \otimes BA \otimes X \to C \otimes BAX$), one finds that

$$\omega_0^B = 1 - \frac{4}{3}\mu_1^2, \qquad \omega_0^C = 1 - \frac{4}{3}\eta_1^2,$$

where $|\mu_0|^2 + |\mu_1|^2 = 1$, and where $|\eta_0|^2 + |\eta_1|^2 = 1$. According to Eq.(16), the following relations hold

$$\mu_1 = \frac{\sqrt{3}\lambda_0 - \lambda_1}{2}, \qquad \eta_1 = -\frac{\sqrt{3}\lambda_0 + \lambda_1}{2}. \tag{58}$$

With a similar reasoning for the map T_1, one finds that

$$\omega_1^A = 1 - \frac{4}{3}(\bar{\lambda}_1^2 + \bar{\lambda}_{1'}^2), \qquad \omega_1^B = 1 - \frac{4}{3}(\bar{\mu}_1^2 + \bar{\mu}_{1'}^2), \qquad \omega_1^C = 1 - \frac{4}{3}(\bar{\eta}_1^2 + \bar{\eta}_{1'}^2), \tag{59}$$

where

$$|\bar{\mu}_0|^2 + |\bar{\mu}_1|^2 + |\bar{\mu}_1'|^2 = 1, \tag{60}$$

$$\begin{aligned}
\bar{\mu}_1 &= \frac{\bar{\lambda}_0 + \bar{\lambda}_{1'}}{\sqrt{2}} & \bar{\mu}_{1'} &= \frac{-\bar{\lambda}_0 + \sqrt{2}\,\bar{\lambda}_1 + \bar{\lambda}_{1'}}{2} \\
\bar{\eta}_1 &= \frac{\bar{\lambda}_0 - \bar{\lambda}_{1'}}{\sqrt{2}} & \bar{\eta}_{1'} &= \frac{-\bar{\lambda}_0 + \sqrt{2}\,\bar{\lambda}_1 - \bar{\lambda}_{1'}}{2}.
\end{aligned} \tag{61}$$

One also easily checks that $\omega_2^A = \omega_2^B = \omega_2^C = -1/3$, so that the map T_2 is useless for optimal cloning, and we can choose $r_2 = 0$. In summary, optimal cloning machines are found after maximizing

$$\begin{aligned}
\omega^A &= r_0 \frac{4\lambda_0^2 - 1}{3} + r_1 \frac{4\bar{\lambda}_0^2 - 1}{3}, \\
\omega^B &= r_0 \frac{(\lambda_0 + \sqrt{3}\lambda_1)^2 - 1}{3} + r_1 \frac{(\bar{\lambda}_0 + \sqrt{2}\,\bar{\lambda}_1 - \bar{\lambda}_{1'})^2 - 1}{3}, \\
\omega^C &= r_0 \frac{(\lambda_0 - \sqrt{3}\lambda_1)^2 - 1}{3} + r_1 \frac{(\bar{\lambda}_0 + \sqrt{2}\,\bar{\lambda}_1 + \bar{\lambda}_{1'})^2 - 1}{3},
\end{aligned} \tag{62}$$

subject to the normalization, and constraints (58),(61).

Numerical calculations suggest that the optimal solution corresponds to $r_1 = 1$ ($r_0 = 0$). In this case, the optimal $1 \to 1 + 2$ machine given above can be recovered taking $\bar{\lambda}_{1'} = 0$. Interestingly, one finds in that case that some quantum information still remains to produce a non-trivial third clone.

Remark: it is very natural to think of using $1 \to 1+1+1$ cloning machines to perform a simultaneous measurement of the three Pauli operators, measuring each Pauli operator at each output of the cloning machine. However such a measurement will not be optimal, as has already been demonstrated in [25] using a symmetric $1 \to 3$ cloning machine.

IV. OPTICAL IMPLEMENTATIONS

We now turn to the issue of implementing some of the machines presented in the previous sections. We will restrict ourselves to optical implementations where qubits are represented by polarisation states of photons: N identical qubit will be represented by N photons in an identical polarisation mode. In the case of symmetric $N \to M$ cloning, such implementations have already been proposed [26], and demonstrated experimentally [27]. Let us first briefly recall how these $N \to M$ cloning machines work. Let

$$|\psi_{\text{in}}\rangle \frac{1}{\sqrt{N!}} (\alpha a_{V,s}^\dagger + \beta a_{H,s}^\dagger)^N |\text{vac}\rangle, \tag{63}$$

denote the input state to clone, where $|\text{vac}\rangle$ denotes the vacuum state. The labels V, H, s stand for vertical, horizontal and signal respectively. Cloning is achieved when a mode prepared in a state (63) impinges a crystal where a parametric down-conversion (PDC) process can occur. The hamiltonian describing this process is of the form:

$$H = \gamma(a_{V,s}^\dagger a_{H,i}^\dagger - a_{H,s}^\dagger a_{V,i}^\dagger) + \text{h.\,c.} \;, \tag{64}$$

where 'i' denotes the idler mode. So the state after the crystal is

$$|\psi_s\rangle \propto e^{-iHt}(\alpha a_{V,s}^\dagger + \beta a_{H,s}^\dagger)^N |\text{vac}\rangle. \tag{65}$$

Looking at those cases where there are M photons in the signal mode, one can see that the optimal fidelities for the $N \to M$ cloning machine for qubits are obtained. Therefore, the successful realization of the cloning machine is conditioned on the number of photons at the output. Note that when M photons are observed in the signal mode, N of them came from the initial state and $M - N$ were produced at the crystal, which means that there are as well $M - N$ photons in the idler mode. These photons are usually called anti-clones. The total number of photons is $2M - N$.

A modification of this scheme was proposed in [28] by Filip, in order to obtain the asymmetric $1 \to 1+1$ machine discussed above. His scheme is shown in Fig. 2a. After a successful $1 \to 2$ cloning (that is, when all the three detectors will click), the two clones are split at the first beam splitter, and one of the clones is combined with the anti-clone at a beam-splitter of transmittivity T. Long but straightforward algebra shows that the fidelities at the modes 1 and 2 give the $1 \to 1+1$ machine, depending on T.
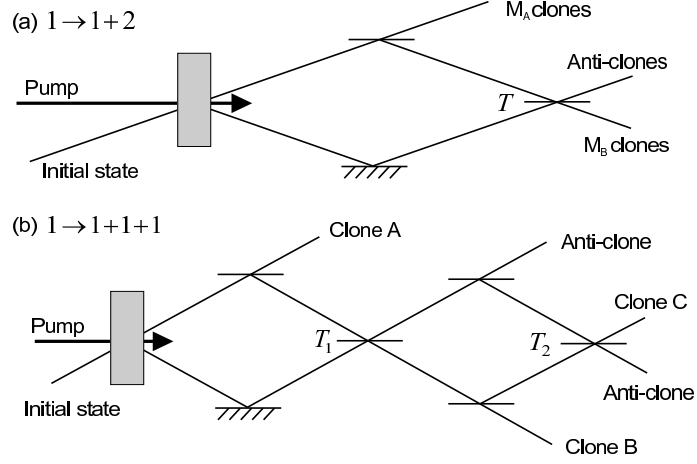
(a) $1 \to 1+2$

M_A clones

Anti-clones

Pump

T

M_B clones

Initial state

(b) $1 \to 1+1+1$

Clone A

Anti-clone

Clone C

Pump

T_1

T_2

Anti-clone

Initial state

Clone B

FIG. 2: Optical implementation of (a) the $1 \to 1+2$ and (b) the $1 \to 1+1+1$ optimal asymmetric cloning machines.

The natural question is now whether this modification also provides the optimal solution for the more general case $N \to M_A + M_B$. Note that the scheme of [26] gives all $N \to M$ machines, simply conditioning on different number of photons at the input and output signal mode. We denote by M_i the number of photons in mode i. As said above, the results of [28] imply that conditioned on the fact that there is one photon at the input signal mode, $N = 1$, and one photon at each output mode, $M_A = M_B = M_a = 1$, the optimal $1 \to 1+1$ machine is obtained. It is also easy to see that this machine is covariant, so all the calculations can be done taking as initial state $(a_{V,s}^\dagger)^N |\text{vac}\rangle$. Then, the state at the output of the crystal when the total number of photons is $2M - N$ reads

$$|\psi_s^M\rangle = \sum_{j=0}^{M-N} (-1)^j \sqrt{\binom{M-j}{N}} |M-j\rangle_{V,s} |j\rangle_{H,s} |j\rangle_{V,i} |M-N-j\rangle_{H,i}. \tag{66}$$

The simplest generalization of the $1 \to 1+1$ result corresponds to $N = 1$, $M_A = 1$ $M_B = 2$. One can check that the evolution of the state (66) through the beam-splitters, where $N = 1$ and $M = M_A + M_B = 3$, gives the following fidelities, depending on T,

$$F_1 = \frac{4T^2 - 4T + 7}{12T^2 - 12T + 9} \qquad\qquad F_2 = F_3 = \frac{8T^2 - 4T + 3}{12T^2 - 12T + 9}. \tag{67}$$

These fidelities are shown in figure 3. Only the relevant part for $1/2 \le T \le 1$ is depicted. Note that for $T = 1$ the optimal symmetric machine is recovered, as expected. If the transmittivity decreases, the quality of the first clone increases, while the quality of the two clones in mode 2 worsens. When $T = 1/2$ all the information on mode 2 (and 3) is lost, and a perfect copy of the initial state is obtained at mode 1.

How can the missing values be obtained? Note that in the previous expressions, the fidelity for the clone in the first mode is always larger than that of the two post-selected photons in mode 2. This suggests a way to find the remaining part of the curve of optimal fidelities: one has to reverse the post-selection of photons, that is look at the cases where $M_A = 2$ and $M_B = 1$. In this way, one expects to reproduce the situation where the fidelity for the two clones, now in mode 1, is larger than the fidelity for the single clone, now in mode 2. Repeating the calculations, but now for $M_A = 2$ and $M_B = 1$, one has

$$F_1 = \frac{7T^2 - 4T + 4}{9T^2 - 12T + 12} \qquad\qquad F_2 = F_3 = \frac{3T^2 - 4T + 8}{9T^2 - 12T + 12}. \tag{68}$$

The corresponding curve is also shown in figure 3. In this way the optimal $1 \to 1+2$ case is completely recovered. Indeed, when $t = 2/3$ it is found that the two photons in mode 1 have fidelities $5/6$, while the photon in mode 2 has fidelity $5/9$, as it should be. Remarkably, the fidelities of Eq. (67) are the same as in Eq. (68), changing T into $1/T$.
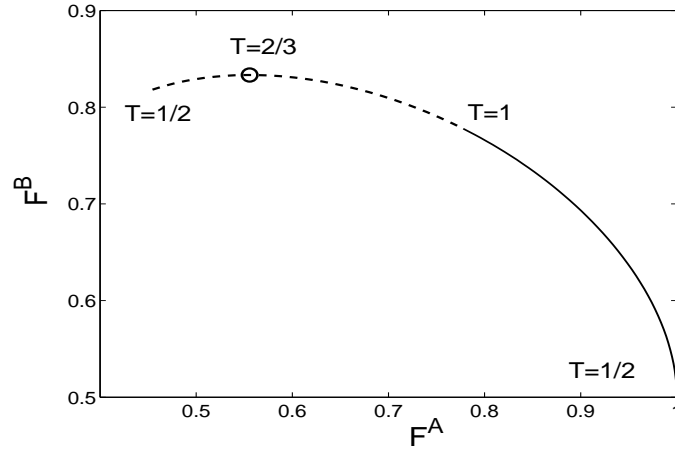
FIG. 3: Clone fidelities for the optical implementation of the $1 \to 1+2$ optimal machine. The solid, resp. dashed, line represents the case when $M_A = 1$ and $M_B = 2$, resp. $M_A = 2$ and $M_B = 1$.

Using the same ideas, we also analyzed the case $2 \to 2+1$. For $M_A = 2$ and $M_B = 1$ one finds, putting $N = 2$ in (66)

$$F_1 = F_2 = 1 - \frac{(2T-1)^2}{4(4T^2 - 4T + 3)} \qquad\qquad F_3 = 1 - \frac{(2T-3)^2}{4(4T^2 - 4T + 3)}, \qquad (69)$$

while for $M_A = 1$ and $M_B = 2$ one has

$$F_1 = F_2 = 1 - \frac{(T-2)^2}{4(3T^2 - 4T + 4)} \qquad\qquad F_3 = 1 - \frac{(3T-2)^2}{4(3T^2 - 4T + 4)}. \qquad (70)$$

Note that, again, (70) can be obtained from (69) if T is replaced by $1/T$. These fidelities are depicted in Fig.**??**, they indeed correspond to the optimal solution given above.

Actually, the case $N \to N + 1$ can also be computed. The obtained fidelities when $M_A = N$ and $M_B = 1$ are

$$F_1 = F_2 = 1 - \frac{(2T-1)^2}{(N+2)(2NT^2 - 2NT + N + 1)} \qquad\qquad F_3 = 1 - \frac{(NT-N+1)^2}{(N+2)(2NT^2 - 2NT + N + 1)}, \qquad (71)$$

while the expressions for $M_A = 1$ and $M_B = N$ are again given from these quantities after replacing T by $1/T$. One can check that the obtained fidelities are identical to Eqs. (47), conjectured to be optimal.

All the previous results give support to the conjecture that all the $N \to M_A + M_B$ cloning machines are included in Filip's scheme, as it happened for the symmetric case [26]. Unfortunately, this is not the case. Indeed, we've checked that this scheme does not provide the optimal solution when $N = 2$, $M_A = 2$ and $M_B = 2$. Therefore, we conjecture that this modification of the symmetric cloning machine implementation only works for the cases $1 \to N + 1$ and $N \to N + 1$, that is when only two irreducible representations appear in the conjectured optimal solution of (6).

This optical scheme can also be adapted to realize the optimal $1 \to 1 + 1 + 1$ cloning transformation, see Fig.2(b). Here, the output of a symmetric $1 \to 3$ machine is made asymmetric by combining some of the clones with anti-clones at two beam splitters, with transmittance T_1 and T_2. The obtained fidelities $F^A \geq F^B \geq F^C$, depending on T_1 and T_2, are optimal. The three fidelities are equal when $T_1 = T_2 = 1$. Other interesting limiting cases are $(F^A, F^B, F^C) = (1, 1/2, 1/2)$ when $T_1 = 1/2$, while taking $T_2 = 1$ gives Eqs. (67) for the $1 \to 1 + 2$ case. This construction can easily be generalized further.

V. CONCLUSIONS

In summary, we have introduced a new class of quantum cloning machines, which, helps to get a better understanding of how quantum information can be distributed unequally between several quantum systems. Mutlipartite asymmetric cloning machines have at least two interesting applications. First, some $N \to M_A + M_B$ machines have been proven to be a useful tool in assessing the security of some quantum cryptographic protocols [13, 14, 15] . Considering

$n \to n+1$ machines, we have seen that if one wants to produce n clones from an input with a fidelity which is as high as possible, some quantum information still remains to produce a non-trivial $n+1$-th clone. Also, we have seen how $1 \to 1+n$ cloning allow, in the limit of large values of n, to study the trade-off between the gain of knowledge about the state of a quantum system and the disturbance undergone by this system. We have also demonstrated feasible optical implementations of some machines. We have seen that the impossibility of perfect cloning translates in the spontaneous emission that unavoidably accompanies stimulated emission.

Several questions remain open. We here list a few of them. First, it would be very desirable to prove the conjecture about the structure of optimal cloning maps (to disprove it would turn even more interesting). Second, it would certainly be interesting to find cloning machines optimal with respect to the global fidelities as defined by Eq.(3), instead of single-copy fidelities as we did in this work. Do optimal machines coincide for both figures of merit as in the case of symmetric cloning [7]? Another interesting problem is to find closed formulas for optimal cloning. We have the feeling that this problem will not be solvable with the techniques presented here. The reason is that our optimisation requires the computation of a so-called Racah coefficient, for which, to our knowledge, no closed formula exists in general. Many interesting questions regarding implementations also remain. It is tempting to believe that all $N \to M_A + M_B + \ldots$ cloning machines are only limited by spontaneous emission, and can therefore be implemented by splitting clones and anticlones produced by stimulated emission using beam splitters. This question deserves further investigation. Finally, it would be interesting to perform an (optical) experiment demonstrating the concepts analysed here.

VI. ACKNOWLEDGEMENTS

APPENDIX A: THE SYMMETRIES OF OPTIMAL CLONING MAPS

We now use the method of Keyl-Werner and exploit symmetries in order to characterize the sought optimal cp-maps T [7]. In a sense this subsection can be considered as a summary of their method. There is however a "twist" with respect to their analysis, due to the fact that there are now two sets of clones with different fidelities.

We will work in Heisenberg picture, where states are left unchanged and operators are transformed. For our purpose, it is convenient to represent the cp-map we look for with the Stinespring dilation theorem [16]. This theorem states that any cloning map can be written as

$$T : \mathcal{B}(\mathcal{H}^{\otimes M}) \to \mathcal{B}(\mathcal{H}_+^{\otimes N}) : \mathcal{O} \to V^*(\mathcal{O} \otimes \mathbf{1}_\mathcal{K})V, \tag{A1}$$

where $\mathbf{1}_\mathcal{K}$ denotes the identity over some auxiliary Hilbert space, $\mathcal{K}$, and where $V : \mathcal{H}_+^{\otimes N} \to \mathcal{H}^{\otimes M} \otimes \mathcal{K}$ is an isometry $(||V\psi|| = ||\psi||)$.

The figure of merit (4) we have chosen is such that the optimal map can of course be sought amongst covariant maps: that is maps T such that $\forall \mathcal{O} \in \mathcal{B}(\mathcal{H}^{\otimes M})$ and $\forall u \in U(d)$,

$$\pi_N^+(u) \, T(\mathcal{O}) \, \pi_N^+(u)^* = T(\pi^{\otimes M}(u) \, \mathcal{O} \, \pi^{\otimes M}(u)^*). \tag{A2}$$

Indeed, since $\forall u \in U(d)$, the 'translated' map $T_u : \mathcal{O} \to \pi_N^+(u)^* \, T(\pi^{\otimes M}(u) \, \mathcal{O} \, \pi^{\otimes M}(u)^*) \, \pi_N^+(u)$ achieves the same fidelity as T, $F^A(T) = F^A(T_u)$. Thus,

$$F^A(T) = \int du F^A(T) = \int du F^A(T_u) \leq F^A(\int du T_u),$$

where du denotes the Haar measure over $U(d)$ [17]($\int du = 1$). Similarly, $F^B(T) \leq F^B(\int du T_u)$. This proves that if we find an optimal cp-map T^{opt}, then $\int du T_u^{\text{opt}}$ is covariant and is optimal too. Thus, searching for an optimal map, we can restrict ourselves to $U(d)$-covariant cp-maps.

This covariance property is the first symmetry property we will use: it merely states that, for the figure of merit we have chosen, no state should be preferred by an optimal cloning machine. Covariance simplifies much the analysis because it allows us to use the covariant form of the Stinespring dilation theorem [16]:

Theorem 1 *If* $T : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ *denotes a cp-map covariant with respect to the representations* $\{U(d) \ni u \rightarrow \pi_1(u) \in \mathcal{B}(\mathcal{H}_1)\}$ *and* $\{U(d) \ni u \rightarrow \pi_2(u) \in \mathcal{B}(\mathcal{H}_2)\}$, *then* T *is of the form (A1) with the auxillary space* $\mathcal{K}$ *being the carrier space of some representation* $\pi_{\mathcal{K}}$ *of* $U(d)$ *and* V *being an "intertwining" operator:*

$$(\pi_1(u) \otimes \pi_{\mathcal{K}}(u))V = V\pi_2(u) \qquad \forall u \in U(d). \tag{A3}$$

The second symmetry property to use is of course permutation invariance. Consider constructing a cloning machine as follows. We apply the cloning machine described by the cp-map T to our input, and then apply some permutation on the output clones of the set A, and some other permutation on the output clones of the set B. Clearly, we expect that the performance of the obtained cloning machine will be the same as that of the cloning machine we started from, whatever the applied permutations are. Let us formalise this property. Let $\mathrm{Sym}(M_A)$ denote the group of permutations of M_A objects and let $\mathrm{Sym}(M_A) \ni p \rightarrow \delta(p)$ denote a representation of $\mathrm{Sym}(M_A)$ acting on $\mathcal{H}^{\otimes M_A}$ as:

$$\delta(p) : \mathcal{H}^{\otimes M_A} \rightarrow \mathcal{H}^{\otimes M_A} : \phi_1 \otimes \ldots \otimes \phi_{M_A} \rightarrow \phi_{p(1)} \otimes \ldots \otimes \phi_{p(M_A)}.$$

For all admissible cloning map T, the permuted map

$$T_p : \mathcal{O} \rightarrow T[(\delta^{-1}(p) \otimes \mathbf{1}^{\otimes M_B})\mathcal{O}(\delta(p) \otimes \mathbf{1}^{\otimes M_B})] \tag{A4}$$

is a cp-map in its own right and yields clones with the same fidelity as T: $F^A(T) = F^A(T_p)$, $\forall p \in S_{M_A}$. The fact that

$$F^A(T) = \frac{1}{M_A!} \sum_{p \in S_{M_A}} F^A(T_p) \leq F^A\left(\frac{1}{M_A!} \sum_{p \in S_{M_A}} T_p\right)$$

implies that searching for an optimal map, we can focus on $\mathrm{Sym}(M_A)$-invariant cp-maps. Similarly, considering the second set of clones, we see that we can focus on $\mathrm{Sym}(M_B)$-invariant cp-maps.

Consider now the restricted map, T^A, obtained upon tracing T over the second set of clones, i.e.

$$T^A : \mathcal{B}(\mathcal{H}^{\otimes M_A}) \rightarrow \mathcal{B}(\mathcal{H}_+^{\otimes N}) : \mathcal{O} \rightarrow T(\mathcal{O} \otimes \mathbf{1}^{\otimes M_B}).$$

Clearly, T^A is a $U(d)$-covariant, $\mathrm{Sym}(M_A)$-invariant map, with range in $\mathcal{B}(\mathcal{H}_+^{\otimes N})$. One can prove that such a map is *non-degenerate* [7], that is there exists a constant $\omega^A(T)$ such that

$$T^A(\partial \pi^{\otimes M_A}(X)) = \omega^A(T)\partial \pi_N^+(X), \tag{A5}$$

for all $X \in \mathrm{su}(d)$. Non-degeneracy of T^A is a manifestation of the "isotropic" nature of the cloning machine. Indeed the fidelity for the clones A reads

$$
\begin{aligned}
F^A(T) &= \frac{1}{d} + \frac{1}{M_A}\mathrm{min}_\psi \langle \psi^{\otimes N}|T^A(\partial \pi^{\otimes M_A}(|\psi\rangle\langle\psi| - \frac{\mathbf{1}}{d}))|\psi^{\otimes N}\rangle \\
&= \frac{1}{d} + \frac{\omega^A(T)}{M_A}\mathrm{min}_\psi \langle \psi^{\otimes N}|T^A(\partial \pi_N^+(|\psi\rangle\langle\psi| - \frac{\mathbf{1}}{d}))|\psi^{\otimes N}\rangle \\
&= \frac{1}{d}(1 + \frac{N}{M_A}\omega^A(T)(d-1))
\end{aligned}
\tag{A6}
$$

The quantity $\frac{N}{M_A}\omega^A(T)$ can be interpreted as the so-called shrinking factor [19]. Clearly, the restricted map T^B associated with the second set of clones is endowed with the same properties as T^A and its cloning quality can also be characterised by some shrinking factor $N\omega^B(T)/M_B$. It is because of non-degeneracy that we said in Sect.II that the clones of each set are charcterised by a single quantity.

We now show permutation invariance allows to decompose the sought cp-map as a convex combination of simpler cp-maps. Let

$$\pi^{\otimes M_A} = \oplus_\alpha \pi_\alpha, \qquad \mathcal{H}^{\otimes M_A} = \oplus_\alpha \mathcal{H}_\alpha, \qquad \mathbf{1}^{\otimes M_A} = \sum_\alpha E_\alpha \tag{A7}$$

denote the decomposition theory of $\pi^{\otimes M_A}$. E_α is the projector onto $\mathcal{H}_\alpha$. Clearly $[E_\alpha, \pi^{\otimes M_A}(u)] = 0$ $\forall u \in \mathrm{U}(d)$. Hence, by Shur's lemma, $T(E_\alpha \otimes \mathbf{1}^{\otimes M_B})$ is proportional to the projector onto $\mathcal{H}_N^+$: $T(E_\alpha \otimes \mathbf{1}^{\otimes M_B}) = r_\alpha S_N$. So

$$T_\alpha : \mathcal{B}(\mathcal{H}_\alpha) \rightarrow \mathcal{B}(\mathcal{H}_N^+) : \mathcal{O} \rightarrow r_\alpha^{-1}T(E_\alpha \otimes \mathbf{1}^{\otimes M_B}\mathcal{O}E_\alpha \otimes \mathbf{1}^{\otimes M_B})$$

is a unital cp-map.

Also, the commutant of (the algebra linearly generated by) all unitaries $\pi^{\otimes M_A}(g)$ is the algebra linearly generated by permutation operators $\delta(p), p \in \mathrm{Sym}(M_A)$.

Clearly, $\forall p \in \mathrm{Sym}(M_A), \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}^{\otimes M_A})$, we have

$$T[\mathcal{O}(\delta(p) \otimes \mathbf{1}^{\otimes M_B})] = T[(\delta(p) \otimes \mathbf{1}^{\otimes M_B})\mathcal{O}].$$

Now since each E_α is a linear combination of permutation operators and since $E_\alpha^2 = E_\alpha$, we have that $T[\mathcal{O}(E_\alpha \otimes \mathbf{1}^{\otimes M_B})] = T[(E_\alpha \otimes \mathbf{1}^{\otimes M_B})\mathcal{O}(E_\alpha \otimes \mathbf{1}^{\otimes M_B})]$. Hence the following decomposition holds

$$T(\mathcal{O}) = \sum_\alpha r_\alpha T_\alpha(\mathcal{O}).$$

Similarly, we can decompose each T_α according to the irreducible representations contained in $\pi^{\otimes M_B}$. Thus, we get

$$T = \sum_{\alpha_1 \in D(M_A)} \sum_{\alpha_2 \in D(M_B)} r_{\alpha_1,\alpha_2} T_{\alpha_1,\alpha_2}, \tag{A8}$$

where $D(M_A) = \{\alpha_1 : \pi_{\alpha_1} \subset \pi^{\otimes M_A}\}$, $D(M_B) = \{\alpha_2 : \pi_{\alpha_2} \subset \pi^{\otimes M_B}\}$ and the coefficients r_{α_1,α_2} are positive constants summing to unity.

It remains to decompose each map T_{α_1,α_2} using the *covariant form* of the Stinespring theorem. The convex decomposition of T_{α_1,α_2} is the same as the reduction theory of $\pi_{\mathcal{K}}$ into irreducibles. Let $\mathbf{1}_{\mathcal{K}} = \sum_\beta \mathbf{1}_\beta$ denote this reduction. $V^*(\mathbf{1}_{\alpha_1} \otimes \mathbf{1}_{\alpha_2} \otimes \mathbf{1}_\beta)V = r_\beta \mathbf{1} \Rightarrow T_{\alpha_1,\alpha_2,\beta} : \mathcal{O} \to r_\beta^{-1} V(\mathcal{O} \otimes \mathbf{1}_\beta)V^*$ is a unital cp-map and the decomposition

$$T_{\alpha_1,\alpha_2} = \sum_\beta r_\beta T_{\alpha_1,\alpha_2,\beta} \tag{A9}$$

holds. In turn, this decomposition induces the decompositions

$$\omega^A = \sum_{\alpha_1,\alpha_2} r_{\alpha_1,\alpha_2} \sum_\beta r_\beta \omega^A_{\alpha_1,\alpha_2,\beta} \tag{A10}$$

$$\omega^B = \sum_{\alpha_1,\alpha_2} r_{\alpha_1,\alpha_2} \sum_\beta r_\beta \omega^B_{\alpha_1,\alpha_2,\beta} \tag{A11}$$

$$\tag{A12}$$

Relation between the fidelities of the clones. We will now characterise the intertwining operator V and see how ω^A and ω^B are related. Addressing the first problem requires that we take care of the *order* in which the decomposition of a representation into irreducible components is carried out (this will be clarified below). Addressing the second problem requires that we can connect these orders of decomposition with each other.

Consider a single map $T_{\alpha_1,\alpha_2,\beta}$, and let us solve the equation

$$V\pi_N^+(g) = (\pi_{\alpha_1}^A(g) \otimes \pi_{\alpha_2}^B(g) \otimes \pi_\beta^X(g))V \tag{A13}$$

We will consider two manners to reduce $\pi_{\alpha_1}^A \otimes \pi_{\alpha_2}^B \otimes \pi_\beta^X$. The first manner first reduces the representation π_{α_2} (associated with the second set of clones, B), with the representation π_β (associated to the auxiliary system, X), and then the resulting representation π_{α_1} (associated with the first set of clones, A):

$$A \otimes B \otimes X \to A \otimes BX \to ABX. \tag{A14}$$

The second manner is:

$$A \otimes B \otimes X \to B \otimes AX \to BAX. \tag{A15}$$

Let us consider the first reduction order. Then

$$\pi_{\alpha_1} \otimes \{\pi_{\alpha_2} \otimes \pi_\beta\} \approx \sum_a \bigoplus_{i_a=1}^{m_a} \pi_{N,i_a}^+ \oplus \pi_{\mathrm{rest}},$$

where m_a denotes the multiplicity of π_N^+ in $\pi_{\alpha_1} \otimes \pi_a$, and where π_{rest} contains no copy of π_N^+. Let us suppose that $m_a \le 1 \ \forall a$. The general case is not more complicated to treat, but this assumption will allow us to adopt lighter notations, and at least it holds in all cases exhibited in this paper. Then, we can rewrite the last equivalence as:

$$\pi_{\alpha_1} \otimes \{\pi_{\alpha_2} \otimes \pi_\beta\} \approx \bigoplus_a \pi_{N,a}^+ \oplus \pi_{\text{rest}}, \tag{A16}$$

and (up to unitaries) V satisfies

$$(\bigoplus_a \pi_{N,a}^+(g) \oplus \pi_{\text{rest}}(g))V = V\pi_N^+(g). \tag{A17}$$

From this relation and Shur's lemma, we infer that there exist coefficients λ_a such that $V = \sum_a \lambda_a V_a$, where $V_a : \mathcal{H}_+^{\otimes N} \to \mathcal{H}_{+,a}^{\otimes N}$ is the unique intertwiner between π_N^+ and $\pi_{N,a}^+$. One can verify the following properties:

$$V_a^* V_b = \delta_{ab} S_N, \tag{A18}$$

$$T_{\alpha_1, \alpha_2, \beta} \text{ is unital} \Rightarrow \sum_a |\lambda_a|^2 = 1,$$

where S_N denotes the identity over $\mathcal{H}_N^+$.

Non-degeneracy of $T_{\alpha_1, \alpha_2, \beta}$ is now expressed as

$$\omega_{\alpha_1, \alpha_2, \beta}^A \partial\pi_N^+(X) = \sum_a |\lambda_a|^2 V_a^* (\partial\pi_{\alpha_1}(X) \otimes \mathbf{1}_a) V_a, \ X \in \text{su}(d). \tag{A19}$$

At this point, it is possible to express ω^A as a function of Casimir numbers $C_2(\pi_a)$, as in [7]. We get

$$\omega_{\alpha_1, \alpha_2, \beta}^A = \frac{1}{2} + \frac{1}{2C_2(\pi_N^+)}\{C_2(\pi_{\alpha_1}) - \sum_a |\lambda_a|^2 C_2(\pi_a)\}. \tag{A20}$$

In this expression, the sum runs over all $a \in D_N(\pi_{\alpha_2} \otimes \pi_\beta) = \{a : \pi_N^+ \subset \pi_{\alpha_2} \otimes \pi_\beta \otimes \pi_a\}$. In the case of qubits (SU(2)), where irreducible representations are labelled by positive half-integer numbers j, we have $C_2(j) = j(j+1)$. Explicit expressions of $C_2(\pi_a)$ for irreducible representations of SU(d) can be found in [7, 17].

What about the second set of clones? Eq.(A20) has been derived following the reduction order (A14), and the fact that $V = \sum_a \lambda_a V_a$. If instead, we had used the reduction order (A15), we would have found that

$$V = \sum_b \mu_b W_b,$$

where $\pi_{\alpha_1} \otimes \pi_\beta \approx \oplus_b \pi_b$, and where W_b intertwines π_N^+ and the (unique) copy of π_N^+ contained in $\pi_{\alpha_2} \otimes \pi_b$ (when any). Thus we get

$$\omega_{\alpha_1, \alpha_2, \beta}^B = \frac{1}{2} + \frac{1}{2C_2(\pi_N^+)}\{C_2(\pi_{\alpha_2}) - \sum_b |\mu_b|^2 C_2(\pi_b)\}. \tag{A21}$$

In this expression, the sum runs over all $b \in D_N(\pi_{\alpha_1} \otimes \pi_\beta)$. All we need now, in order to quantify the trade-off between the qualities of the two sets of clones, is a relation between the coefficients λ_a and the coefficients μ_b. It is easy to find such a relation: just observe that $V = \sum_a \lambda_a V_a = \sum_z \mu_z W_z \Rightarrow \sum_a \lambda_a W_b^* V_a = \sum_z \mu_z W_b^* W_z = \mu_b S_N$. Thus

$$\mu_b = \frac{1}{\dim\mathcal{H}_+^{\otimes N}} \sum_a \lambda_a \operatorname{tr} W_b^* V_a. \tag{A22}$$

N.B. The quantity $\operatorname{tr} W_b^* V_a$ is known in representation theory as the Racah coefficient.

APPENDIX B: CALCULATIONS RELATED TO $1 \to 1+n$ CLONING OF QUBITS

We are looking for a map $T : \mathcal{B}(\mathcal{H} \otimes \mathcal{H}^{\otimes n}) \to \mathcal{B}(\mathcal{H})$. According to Section II C, T decomposes as

$$T = \sum_{\alpha_2 \in D(\pi^{\otimes n})} \sum_{\beta \in D_1(\pi_{1/2} \otimes \pi_{\alpha_2})} r(\alpha_2, \beta) T(\alpha_2, \beta).$$

$D(\pi^{\otimes n})$ is given by the decomposition theory of $\pi_{1/2}^{\otimes n}$, which is well-known [17]:

$$\pi_{1/2}^{\otimes n} = \sum_{s \in I_n} \bigoplus_{i_s=1}^{m_s} \pi_{s,i_s}$$

In this expression, m_s denotes the multiplicity of the representation π_s, and $I_n = \{1/2, \dots, n/2\}$ when n is odd, and $I_n = \{0, \dots, n/2\}$ when n is even. Accordingly,

$$T = \sum_{s \in I_n} \sum_{i_s=1}^{m_s} r(s, i_s) T(s, i_s),$$

where $T(s, i_s) : \mathcal{B}(\mathcal{H}_{1/2} \otimes \mathcal{H}_{s,i_s}) \to \mathcal{B}(\mathcal{H}_{1/2})$, $r(s, i_s) \geq 0$ and $\sum_{s,i_s} r(s, i_s) = 1$. Before decomposing the map T any further, we remark that, for fixed s, all maps $T(s, i_s)$ are isomorphic. Therefore, as far as optimality is concerned, families of cloning machines with the same values of $r_s = \sum_{i_s} r(s, i_s)$ are equivalent. This fact allows to get rid of the multiplicities of each s and simply write

$$T = \sum_{s \in I_n} \sum_{\beta \in D_1(\pi_{1/2} \otimes \pi_s)} r(s, \beta) T(s, \beta). \tag{B1}$$

Let us characterise $D_1(\pi_{1/2} \otimes \pi_s)$. We have $\pi_{1/2} \subset \pi_{1/2} \otimes \pi_s \otimes \pi_\beta \iff \pi_\beta \subset \pi_{1/2}^{\otimes 2} \otimes \pi_s$. There are three cases to consider: Case A: $s = 0$, which yields $D_1(\pi_{1/2} \otimes \pi_0) = \{0, 1\}$. This case occurs whenever n is even. Case B: $s = 1/2$; which yields $D_1(\pi_{1/2} \otimes \pi_{1/2}) = \{1/2, 3/2\}$. This case occurs whenever n is odd. Case C: $s > 1/2$, which yields $D_1(\pi_{1/2} \otimes \pi_s) = \{s-1, s, s+1\}$. This case occurs whenever $n > 1$.

Let us start with case A. So suppose that in the convex decomposition of the cloning map, T, some map $T(0) : \mathcal{B}(\mathcal{H}_{1/2} \otimes \mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_{1/2})$ appears. $T(0)$ decomposes as $T(0) = r(0,0) T(0,0) + r(0,1) T(0,1)$. $T(0,0)$ and $T(0,1)$ have the following structure:

$$T(0,0) \; : \; \mathcal{O} \to V(0,0)^*(\mathcal{O} \otimes \mathbf{1}_0) V(0,0), \tag{B2}$$
$$T(0,1) \; : \; \mathcal{O} \to V(0,1)^*(\mathcal{O} \otimes \mathbf{1}_1) V(0,1), \tag{B3}$$

where $(\pi_{1/2}^A \otimes \pi_0^B \otimes \pi_0^X) V(0,0) = V(0,0) \pi_{1/2}$, and $(\pi_{1/2}^A \otimes \pi_0^B \otimes \pi_1^X) V(0,1) = V(0,1) \pi_{1/2}$. Thus,

$$\omega^A(1/2, 0, 0) \; = \; \frac{1}{2}\{1 + \frac{C_2(1/2) - C_2(0)}{C_2(1/2)}\} = 1, \qquad \omega^B(1/2, 0, 0) = \frac{1}{2}\{1 + \frac{C_2(0) - C_2(1/2)}{C_2(1/2)}\} = 0, \tag{B4}$$

$$\omega^A(1/2, 0, 1) \; = \; \frac{1}{2}\{1 + \frac{C_2(1/2) - C_2(1)}{C_2(1/2)}\} = -1/3, \qquad \omega^B(1/2, 0, 1) = \frac{1}{2}\{1 + \frac{C_2(0) - C_2(1/2)}{C_2(1/2)}\} = 0. \tag{B5}$$

We see that the map $T(0,1)$ is useless for cloning.

The case B is straightforward to treat. Suppose now that in the decomposition of T into irreducible summands, a map $T(1/2, 1/2) : \mathcal{B}(\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}) \to \mathcal{B}(\mathcal{H}_{1/2})$ appears. This map decomposes as $T(1/2) = r(1/2, 1/2) T(1/2, 1/2) + r(1/2, 3/2) T(1/2, 3/2)$. The maps $T(1/2, 1/2)$ and $T(1/2, 3/2)$ are exactly those encountered in $1 \to 1+1$ cloning. Thus we see immediately from the results of Section III A that:

$$\omega^A(1/2, 1/2) \; = \; 1 - \frac{4}{3}\lambda(1/2, 1/2)^2, \tag{B6}$$

$$\omega^B(1/2, 1/2) \; = \; 1 - \frac{4}{3}\mu(1/2, 1/2)^2 = 1 - \frac{4}{3}(\frac{\sqrt{3}}{2}\sqrt{1 - \lambda(1/2, 1/2)^2} - \frac{1}{2}\lambda(1/2, 1/2))^2, \tag{B7}$$

where $0 \geq |\lambda(1/2, 1/2)| \geq 1$, and that the map $T(1/2, 3/2)$ is useless for cloning.

We now turn to case C. The convex decomposition of the cloning map T now contains terms $T(s) : \mathcal{B}(\mathcal{H}_{1/2} \otimes \mathcal{H}_s) \to \mathcal{B}(\mathcal{H}_{1/2})$. Each of these maps $T(s)$ decomposes as $T(s) = r(s, s-1) T(s, s-1) + r(s, s) T(s, s) + r(s, s+1) T(s, s+1)$.

Each map $T(s, s-1)$ reads

$$T(s, s-1) : \mathcal{O} \rightarrow V(s, s-1)^*(\mathcal{O} \otimes \mathbf{1}_{s-1})V(s, s-1), \tag{B8}$$

where

$$(\pi^A_{1/2} \otimes \pi^B_s \otimes \pi^X_{s-1})V(s, s-1) = V(s, s-1)\pi_{1/2}. \tag{B9}$$

There exists a unitary Clebsch-Gordan matrix C such that

$$(\pi^A_{1/2} \otimes (\pi^{BX}_1 \oplus \ldots))CV(s, s-1) = CV(s, s-1)\pi_{1/2}.$$

We deduce that

$$\omega^A(s, s-1) = \frac{1}{2}\{1 + \frac{C_2(1/2) - C_2(1)}{C_2(1/2)}\} = -1/3. \tag{B10}$$

There also exists a unitary Clebsch-Gordan matrix D such that

$$(\pi^B_1 \otimes \pi^{AX}_{1/2})DV(s, s-1) = DV(s, s-1)\pi_{1/2},$$

if $s = 1$, whereas Eq.(B9) imply that

$$(\pi^B_1 \otimes (\pi^{AX}_{s-1/2} \oplus \pi^{AX}_{s-3/2}))DV(s, s-1) = DV(s, s-1)\pi_{1/2},$$

for $s \geq 1$. We infer that

$$\omega^B(s, s-1) = \frac{2}{3}(s+1). \tag{B11}$$

Let us now consider the maps $T(s, s)$. Each such map reads

$$T(s, s) : \mathcal{O} \rightarrow V(s, s)^*(\mathcal{O} \otimes \mathbf{1}_s)V(s, s), \tag{B12}$$

where

$$(\pi^A_{1/2} \otimes \pi^B_s \otimes \pi^X_s)V(s, s) = V(s, s)\pi_{1/2}. \tag{B13}$$

Again, there exists some unitary Clebsch-Gordan matrix, which we denote again C, such that

$$(\pi^A_{1/2} \otimes (\pi^{BX}_0 \oplus \pi^{BX}_1 \oplus \ldots))CV(s, s) = CV(s, s)\pi_{1/2}.$$

From Shur's lemma, $V(s, s)$ decomposes as $V(s, s) = \lambda(s, s, 0)V(s, s, 0) + \lambda(s, s, 1)V(s, s, 1)$, where $|\lambda(s, s, 0)|^2 + |\lambda(s, s, 1)|^2 = 1$, where $V(s, s, 0)$ intertwines $\pi_{1/2}$ with the (unique) copy of $\pi_{1/2}$ contained in $\pi^A_{1/2} \otimes \pi^{BX}_0$ and where $V(s, s, 1)$ intertwines $\pi_{1/2}$ with the (unique) copy of $\pi_{1/2}$ contained in $\pi^A_{1/2} \otimes \pi^{BX}_1$. Accordingly, we find that

$$\omega^A(s, s) = \frac{1}{2}\{1 + \frac{C_2(1/2) - (|\lambda(s, s, 0)|^2 C_2(0) + |\lambda(s, s, 1)|^2 C_2(1))}{C_2(1/2)}\} = 1 - \frac{4}{3}|\lambda(s, s, 1)|^2, \tag{B14}$$

A similar reasoning considering the second set of clones gives $V(s, s) = \mu(s, s, s-1/2)W(s, s, s-1/2) + \mu(s, s, s+1/2)W(s, s, s+1/2)$, where $|\mu(s, s, s-1/2)|^2 + |\mu(s, s, s+1/2)|^2 = 1$, where $W(s, s, s-1/2)$ intertwines $\pi_{1/2}$ with the (unique) copy of $\pi_{1/2}$ contained in $\pi^B_s \otimes \pi^{AX}_{s-1/2}$ and where $V(s, s, s+1/2)$ intertwines $\pi_{1/2}$ with the (unique) copy of $\pi_{1/2}$ contained in $\pi^B_s \otimes \pi^{AX}_{s+1/2}$. Accordingly,

$$\omega^B(s, s) = \frac{1}{2}\{1 + \frac{C_2(s) - (|\mu(s, s, s-1/2)|^2 C_2(s-1/2) + |\mu(s, s, s+1/2)|^2 C_2(s+1/2))}{C_2(1/2)}\},$$

The intertwiners V's and W's are explicitly given by

$$V(s, s, 0) = C^{(1/2u)(00)}_{(1/2m)} C^{(sv)(sw)}_{(00)} |1/2, u\rangle_A |s, v\rangle_B |s, w\rangle_X \langle 1/2, m|, \tag{B15}$$

$$V(s, s, 1) = C^{(1/2u)(1l)}_{(1/2m)} C^{(sv)(sw)}_{(1l)} |1/2, u\rangle_A |s, v\rangle_B |s, w\rangle_X \langle 1/2, m|, \tag{B16}$$

$$W(s, s, s-1/2) = C^{(sv)(s-1/2z)}_{(1/2m)} C^{(1/2u)(sw)}_{(s-1/2z)} |1/2, u\rangle_A |s, v\rangle_B |s, w\rangle_X \langle 1/2, m|, \tag{B17}$$

$$W(s, s, s+1/2) = C^{(sv)(s+1/2y)}_{(1/2m)} C^{(1/2u)(sw)}_{(s+1/2y)} |1/2, u\rangle_A |s, v\rangle_B |s, w\rangle_X \langle 1/2, m|. \tag{B18}$$

The following relations hold between the λ's and the μ' s:

$$\mu(s,s,s-1/2) = \frac{1}{2}\{\lambda(s,s,0)\operatorname{tr} W(s,s-1/2)^*V(s,s,0) + \lambda(s,s,1)\operatorname{tr} W(s,s,s-1/2)^*V(s,s,1)\} \tag{B19}$$

$$\mu(s,s,s+1/2) = \frac{1}{2}\{\lambda(s,s,0)\operatorname{tr} W(s,s,s+1/2)^*V(s,s,0) + \lambda(s,s,1)\operatorname{tr} W(s,s,s+1/2)^*V(s,s,1)\}. \tag{B20}$$

From an explicit calculation(using Mathematica), one gets:

$$\operatorname{tr} W(s,s,s-1/2)^*V(s,s,0) = -\operatorname{tr} W(s,s,s+1/2)^*V(s,s,1) = 2\sqrt{\frac{s}{2(s+1/2)}}, \tag{B21}$$

$$\operatorname{tr} W(s,s,s-1/2)^*V(s,s,1) = \operatorname{tr} W(s,s,s+1/2)^*V(s,s,0) = \sqrt{\frac{2(s+1)}{s+1/2}}. \tag{B22}$$

From which we find

$$\omega^B(s,s) = \frac{2}{3}(1-|\lambda(s,s,0)|^2) + \frac{4}{3}\lambda(s,s,0)\lambda(s,s,1)\sqrt{s(s+1)}.$$

We now turn to the third and last piece: the maps $T(s,s+1): \mathcal{B}(\mathcal{H}_{1/2}\otimes\mathcal{H}_s) \to \mathcal{B}(\mathcal{H}_{1/2})$. With a reasoning similar to the analysis of the maps $T(s,s-1)$ and $T(s,s)$, one finds that

$$\omega^A(s,s+1) = \frac{1}{2}\{1 + \frac{C_2(1/2)-C_2(1)}{C_2(1/2)}\} = -1/3.$$

and

$$\omega^B(s,s+1) = \frac{1}{2}\{1 + \frac{C_2(s)-C_2(s+1/2)}{C_2(1/2)}\} = -\frac{2}{3}s.$$

So, we see that the maps $T(s,s+1)$ are useless for cloning.

Extremisation. Let us first consider the case where n is even. Optimal cloning maps are of the form

$$T = r(0,0)T(0,0) + \sum_{s=1}^{n/2}(r(s,s-1)T(s,s-1) + r(s,s)T(s,s)).$$

Note that $\omega^A(s,s-1) = -1/3\ \forall s \geq 1$ and that $\omega^B(s,s-1)$ strictly increases with s. Thus, for the sake of optimality, we can choose $r(s,s-1) = 0\ \forall s < n/2$. Also, one can see that optimal maps can be found for $\lambda(s,s,0), \lambda(s,s,1) \geq 0$ and for $r(s,s) = 0\ \forall s < n/2$. Introducing lighter notations; $r(0,0) \equiv a$, $r(n/2,n/2-1) \equiv b$, $r(n/2,n/2) \equiv c$, $\lambda(n/2,n/2,0) \equiv x$, and $\lambda(n/2,n/2,1) \equiv y$, and extremising, we get Eqs.(49).

A similar argument holds when n is odd ($n \geq 3$) and also leads to Eqs.(49).

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